The Xmath ODE algorithm

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Abstract—*The Xmath eBook is being developed and algorithms into a wide range of undergraduate mathematical issues embeded in Mathematica packages are available on the web using the system webMathematica. The main purpose is to visualize mathematics in the same way as would a professor do it on the blackboard stating all intermediate steps for user defined input and then presenting solutions being easily recognized by the undegraduate student which may not always be the case using the Mathematica system directly. In this way the student may work more on a personal basis, viewing one step at a time in the solving process and then being less dependent of the professors. In this paper the Xmath algorithms for solving Ordinary Differential Equations step-by-step are presented (ODE Steplet).*

Index Terms— *Ordinary Differential Equation (ODE), Steplet, Mathematica packages, Online calculations, Pedagogical value*

1. Introdction

The use of *Mathematica* [1] in education is one of the most important areas of application. The problem however is that in education we are focusing on how problems are solved perhaps more than on the final result. Since *Mathematica* only gives the final result it will be necessary to build an application on top of *Mathematica* giving intermediate results using the methods of solving given by mathematical textbooks. It is necessary to analyze the equations in depth, Xmath then using the *Mathematica* object TreeForm to be able to extract the information needed at each level of the solution process. The algorithm is different from the algorithm used by the developers of the *Mathematica* System (DSolve).

The Xmath algorithm will solve problems typical in mathematical teaching. Linear first and second order ODE, separable equations and suitable substitutions are implemented to make some equations important in education separable, the reduction to linear form for non-linear 1st order ODE is implemented and of course exact differential equations. The main purpose is not to do the algorithm as general as possible like DSolve but to track the solving process in detail to be easily recognized by the students for most problems in particular undergraduate education.

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2. Pedagogical value

The pedagogical value of the Xmath algorithms lies in the fact that a student may simulate solving by changing parameters and type of equation. In my own class last year the students had to solve linear second order ODE with constant coefficients for different values of parameters. This was a success. The important thing is that Xmath solves the equations as would a professor do it on the blackboard then easily being recognized by the students which is not the case using the *Mathematica* system directly [2].

3. Ordinary differential equations (ODE)

ODE's are differential equations in one free variable not including partial derivatives [3]. Given the differential equation

$$(1) \quad y'' + 3y' = 3\cosh(x)$$

The equation may be solved by *Mathematica* using the built in objects DSolve and FullSimplify, the ouput being visually different from the output given by the Xmath algorithm. Xmath will also give all intermediate solutions and the methods for finding them. The Xmath algorithm will give an answer more in line with standard textbooks and blackboard teaching easily recognizing the homogenous and particular parts. Solution given by Xmath:

$$(2) \quad y(x) = c_1 + c_2 e^{-3x} - \frac{1}{2} x e^{-3x} + \frac{1}{12} e^{3x}$$

4. Second order linear ODE with constant coefficients

Second order equations with constant coefficients are given in the general form [4]

$$(3) \quad ay'' + by' + cy = f(x)$$

The eqaution is broken down for analyzing by using the *Mathematica* object TreeForm with 3 levels. The *Mathematica* object FreeQ is used to verify independence of the free variable and this gives the pseudocode for the homogenous part y_H.

```
k=Reverse[Level[TreeForm[ay''+by'+cy==0],3]];
a=k[[4]]/y'';b=k[[5]]/y';c=k[[6]]/y;
If [FreeQ[a,x]&&FreeQ[b,x]&&FreeQ[c,x],
        solvlist=Solve[c+bλ+aλ^2==0];
        (* 4 cases*)
        Which[a===0,yH=C[1] E^(λx),
                b^2-4ac>0, yH=C[1]*E^(λ1x)+C[2]*E^(λ2 x),
                b^2-4ac<0,
yH=E^(Re[λx](C[1]Sin[Im[λ]x]+C[2]Cos[Im[λ]x])
        (*end Which*)]
] (*end If*)
```

Figure 1 PseudoCode Homogenous part

The particular solution y_P of (3) is found by choosing a trial solution in accordance with the right hand side f(x). The built in *Mathematica* Object PolynomialQ is used and further the Xmath Objects ExpQ , ExpPolyQ , TrigQ, TrigPolyQ, TrigExpQ and TrigExpPolyQ . The trial solution will be multiplied with x or x^2 if parts of y_P are common with parts of the homogenous solution y_H. The algorithm for finding y_P is also using the *Mathematica* objects AppendTo, Expand , CoefficientList and the general method Undetermined Coefficients developed in Xmath. The built in *Mathematica* object ReplaceAll (/.) is also used.

```
k=Reverse[Level[TreeForm[ay''+by'+cy==f[x]],3]];
If[Different,kP={function1,function2,..},kP={rhs}];
(*Different gives TRUE for a sum of different kinds of functions
composing rhs, right hand side*)
yPtotal=0;
Do[
        Which[PolynomialQ[kP[[i]],x],yP=∑x^i Ai,
                ExpQ[kP[[i]],x],yP=E^(αx)A[1],
                ExpPolyQ[kP[[i]],x],yP=E^(αx)∑ x^i Ai,
                TrigQ[kP[[i]],x],yP=A[1]Sin[βx]+A[2]Cos[βx],
                TrigPolyQ[kP[[i]],x],yP=Sin[βx] ∑ x^i Ai
+Cos[βx] ∑x^i Bi,
                TrigExpPolyQ[kP[[i]],x],yP= E^(αx) Sin[βx] ∑
x^i Ai +
                        E^(αx) Cos[βx] ∑x^i Bi,
                Head[rhs]==Sec, Using variation of
parameters,
                (* Csc,,Tan and Cot accordingly*)
        (*end Which*) ]
(*using undetermined coefficients  to determine A[i]*)
unknowns=A[i]; SortyP=0;eqnList={};
eq=eq/.{y''->D[D[yP,x],x],y'->D[yP,x],y->yP};
Do[AppendTo[eqnlist,CoefficientList[eq[[j]]==0];
         SortYp≡0;            (*identical powers of x*)
         solvlist=Solve[eqnlist,unknowns];
         yP=Expand[yP/.solvlist;
         yPtotal=yPtotal+yP,{I,1,Length[kP]}] (*end Do*)
y[x]=yH+Yptotal;
```

Figure 2 PseudoCode Particular part

5. SEPARABLE EQUATIONS

A general first order equation is given as

$$(4) \quad h(y, y', x) = 0$$

Xmath solves first order separable equations [5] which might be written in the form

$$(5) \quad y' = \frac{g(x)}{f(y)}$$

Some equations which are not separable may be transformed to a separable equation by using a suitable substitution. These kinds of equations are in Xmath called pseudo separable equations and the substitution implemented is y→xy giving

$$(6) \quad h(y, y', x) = 0 /.\{y \to xy, y' \to y + xy'\}$$

Using the built in *Mathematica* object Integrate and the Xmath object xSeparate we may solve equation (5).

```
solvlist=Solve[h[y.y',x]==0,{y'}]; (*y'→p[x,y]*)
(*separating x and y expressions*)
g[x]=xSeparate[p[x,y],x]; f[y]=1/xSeparate[p[x,y],y];
If[p[x,y]===g[x]/f[y],
        y[x]=Solve[Integrate[f[y],y]==Integrate[g[x],x],y];
        If[substitution,y→xy,y], Not separable
] (*end If*)
```

Figure 3 PseudoCode Separable equation

6. INTEGRATING FACTOR

This method applies to linear 1st order equations but the coefficients need not to be constants [6]. The eqautions is given as

$$(7) \quad y \cdot p(x) + y' = r(x)$$

If p(x)=0 or r(x)=0 the equation is separabel. Xmath will reorganize (7) as ay'+by+c=0 and TreeForm is used to break the equation down.
The method for solving (7) defines a new function v(x) by

$$(8) \quad y(x) = e^{-\int p(x)dx} v(x)$$

The solution is given by

$$(9\ ) \quad y(x) = e^{-\int p(x)dx} \int e^{\int p(x)dx} r(x)dx$$

```
k=Reverse[Level[TreeForm[ay'+by+c==0],3]];
a=k[[4]]/y'; b=k[[5]]/y;c=k[[6]];
p[x]=b/a;r[x]=-c/a;
IntFactor=Integrate[E^(-p[x])],x;
Solve[IntFactor*v'[x]==r[x],v'[x]];
v[x]=Integrate[v'[x],x];
y[x]=Intfactor*v[x];
```

Figure 4 PseudoCode Integrating factor

7. REDUCTION TO LINEAR FORM

Given a non-linear equation of the form, a any real number

$$(10) \quad y \cdot P(x) + y' = y^a Q(x)$$

The equation is called a Bernoulli equation [7] and may be reduced to linear form by using the substitution $u=y^{1-a}$, effectively

$$(11) \quad y \to y^{\frac{1}{1-a}},\ y' \to \frac{y^{\frac{1}{1-a}-1} y'}{1-a}$$

This will give the linear equation

$$(12) \quad (1-a) \cdot y \cdot P(x) + y' = (1-a) Q(x)$$

Equation (12) is of the form (7) and may be solved using integrating factor, substituting back

$$(13) \quad y = y /. \{ y \to y^{\frac{1}{1-a}} \}$$

8. EXACT DIFFERENTIAL EQUATIONS

This method applies to equations of the form [8]

$$(14) \quad M(x,y) + N(x,y) \cdot y' = 0$$

The exact (total) differential of a function u(x,y) is given by

$$(15) \quad du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy$$

Applying (14) this gives

$$(16) \quad du = M(x,y)dx + N(x,y)dy = 0$$

The implicit solution of (16) is u(x,y)=constant and the problem will then be to find u(x,y). We then have

$$(17) \quad \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x} \Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

```
k=Reverse[Level[TreeForm[M(x,y)+N(x,y)*y'==0],3]];
M[x,y]=k[[5]]; N[x,y]=k[[4]]/y';
If [        D[M[x,y],y]===D[N[x,y],x],
            u[x,y]=Integrate[M[x,y],x]+f[y] (*integration constant*)
            f ' [y]=Solve[D[u[x,y],y]==N[x,y],f'[y]];
            f[y]=Integrate[f ' [y],y];
            (*u[x,y]=c, constant,du=0*)
            y[x]=Solve[u[x,y]==c,y],
            (*else*) "Not exact"
] (*end If*)
```

Figure 5 PseudoCode Exact Equations

9. VARIATION OF PARAMETERS

The method variation of parameters [9] applies to liner second order equations and the coefficients need not to be constant. Xmath however will only solve equations with constant coefficients and use variation of parameters in the case where no trial solution is available, f.x in (3), f(x) = sec(x). For solving using variation of parameters, (3) has to be rewritten

$$(18) \quad \frac{cy}{a} + \frac{by}{a} + y'' = \frac{f(x)}{a} = r(x)$$

The homogenous solution is given by $y_H = c_1 y_1 + c_2 y_2$ and we define the Wronskian as $W=-y_2 y_1' + y_1 y_2'$ giving the particular solution

$$(19) \quad y_p = \int \frac{y_1 r(x)}{W} dx \cdot y_2 - \int \frac{y_2 r(x)}{W} dx \cdot y_1$$

The integrations may give complicated results or no results at all especially when the coefficients are not constants (not implemented in Xmath). Equations which theoretically may be solved by this method will often have to be solved using numerical methods implemented in the *Mathematica* object NDSolve.

```
y[1]=yH[1]/.C[1]->1;
y[2]=yH[2]/.C[2]->1;
W=-y[2]*y[1]'+y[1]*y[2]';
r[x]=rhs/a;
yP=-y[1]*Integrate[y[2]*r[x]/W,x]+
y[2]*Integrate[y[1]r[x]/W,x]//Simplify;
y[x]=yH+yP;
```

Figure 6 PseudoCode Variation of Parameters

10. ODE INITIAL VALUE PROBELMS

For second order equations we need to know the values of y(x) and y'(x) for a given value of x (x0). The constants C[i] may then be found. For 1st order the value of y(x) at a given value of x is sufficient. If infinite values appear or if C[i] cannot be found for other reasons a meassage is given and the initial values have to be changed.

11. EXAMPLE

The Xmath algorithms may solve an arbitrary ODE problem in the areas implemented or give a message that Xmath cannot solve the problem in question. In this case calling for the *Mathematica* solution based on generic kinds of algorithms only giving the final answer.

Solve the equation

$$x^3+3y^2x+(y^3+3x^2y)\cdot y'=0$$

Task 1

Type of equation

First order, includes y'(x) as highest derivative
Substitution: y→x y gives separable equation
This gives: y'→(xy)' = y+xy'
May be written in the form f(y) dy = g(x) dx
Finding y(x) by integrating: ∫f(y)dy=∫g(x)dx

Task 2

Finding y'(x)

Given: $x^3+3y^2x+(y^3+3x^2y)\cdot y'=0$

Not separable

Substitution y→x y gives

$$x^3(y^4+6y^2+x(y^2+3)y'y+1)=0$$

$$y'=\frac{dy}{dx}=\frac{-y^4-6y^2-1}{xy(y^2+3)}$$

Task 3

Separating variables

$$\frac{dy}{dx}=\frac{-y^4-6y^2-1}{xy(y^2+3)}$$

$$\frac{y(y^2+3)}{-y^4-6y^2-1}\frac{dy}{dx}=\frac{1}{x}$$

$$\frac{y(y^2+3)}{-y^4-6y^2-1}dy=\frac{1}{x}dx$$

Task 4

Integrating the equation

$$\int\frac{y(y^2+3)}{-y^4-6y^2-1}dy=\int\frac{1}{x}dx$$

$$-\frac{1}{4}\log(y^4+6y^2+1)=c+\log(x)$$

Task 5

Solving for y(x)

Real parts:

$$\{y\to-\sqrt{\frac{c\sqrt{x^8+x^4}}{x^4}-3},\ y\to\sqrt{\frac{c\sqrt{x^8+x^4}}{x^4}-3}\}$$

Substituting back: y→x·y

$$y_1(x)=-\sqrt{\sqrt{8x^4+c}-3x^2}\ ,\ y_2(x)=\sqrt{\sqrt{8x^4+c}-3x^2}$$

Task 6

Implementing initial values

y(0)=1

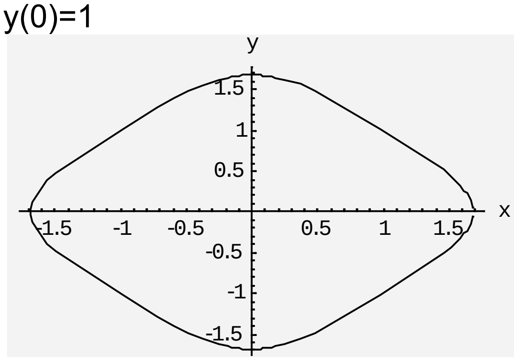
Solving for c: {c→1}

$$y_1(x)=-\sqrt{\sqrt{8x^4+1}-3x^2}\ ,\ y_2(x)=\sqrt{\sqrt{8x^4+1}-3x^2}$$

Task 7

Graphics

y(0)=1



ACKNOWLEDGMENT

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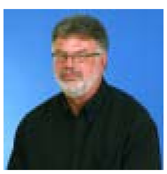

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